

### Exercise 3

Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \dots + \theta_n)} \quad (n = 2, 3, \dots).$$

#### Solution

Start by showing that the result is true in the base case  $n = 2$ .

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Now assume the inductive hypothesis,

$$e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_k} = e^{i(\theta_1 + \theta_2 + \dots + \theta_k)} \quad (k = 2, 3, \dots).$$

The aim is to show that

$$e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_k} e^{i\theta_{k+1}} = e^{i(\theta_1 + \theta_2 + \dots + \theta_k + \theta_{k+1})} \quad (k = 2, 3, \dots).$$

We have

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_k} e^{i\theta_{k+1}} &= (e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_k}) e^{i\theta_{k+1}} \\ &= [e^{i(\theta_1 + \theta_2 + \dots + \theta_k)}] e^{i\theta_{k+1}} \\ &= [\cos(\theta_1 + \theta_2 + \dots + \theta_k) + i \sin(\theta_1 + \theta_2 + \dots + \theta_k)] (\cos \theta_{k+1} + i \sin \theta_{k+1}) \\ &= \cos(\theta_1 + \theta_2 + \dots + \theta_k) \cos \theta_{k+1} - \sin(\theta_1 + \theta_2 + \dots + \theta_k) \sin \theta_{k+1} \\ &\quad + i[\cos(\theta_1 + \theta_2 + \dots + \theta_k) \sin \theta_{k+1} + \sin(\theta_1 + \theta_2 + \dots + \theta_k) \cos \theta_{k+1}] \\ &= \cos[(\theta_1 + \theta_2 + \dots + \theta_k) + \theta_{k+1}] + i \sin[(\theta_1 + \theta_2 + \dots + \theta_k) + \theta_{k+1}] \\ &= \cos(\theta_1 + \theta_2 + \dots + \theta_k + \theta_{k+1}) + i \sin(\theta_1 + \theta_2 + \dots + \theta_k + \theta_{k+1}) \\ &= e^{i(\theta_1 + \theta_2 + \dots + \theta_k + \theta_{k+1})}. \end{aligned}$$

Therefore, by mathematical induction,

$$e^{i\theta_1} e^{i\theta_2} \dots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \dots + \theta_n)} \quad (n = 2, 3, \dots).$$